COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

ELASTOPLASTICITY WITH DAMAGE CONSIDERATIONS IN THE ANALYSIS OF THICK PLATES

João Carlos Arantes Costa Júnior
Universidade Federal de Santa Catarina, Depto. de Engenharia Mecânica
Cx. P. 476 – 88010-970 – Florianópolis, SC, Brasil
Marcelo Krajnc Alves
Universidade Federal de Santa Catarina, Depto. de Engenharia Mecânica
Cx. P. 476 – 88010-970 – Florianópolis, SC, Brasil

Abstract. *The objective of this work is to present a theory and to propose a numerical scheme for the elastoplastic with damage analysis of thick plates, subjected to a cyclic loading. The theory is based on the work done by Lemaitre (1992) and it is based on the thermodynamics of irreversible processes. The elastoplastic model considers a nonlinear isotropic and cinematic hardening law, which depends on the accumulate plastic deformation and presents a saturation level in order to characterize a stabilization of the hardening phenomena. Moreover, the model considers the damage to be isotropic. With the objective of solving thick plate problems, we make use of a higher order plate theory. The discretization of the cyclic elastoplastic with damage problem employs: the Galerkin finite element method, for the discretization of the geometric domain; and the generalized trapezoidal integration rule for the time evolution process. In order to integrate the local evolution equations we make use of an "elastic with damage predictor with a plastic corrector scheme". This algorithm, presented by Simo & Taylor (1986) and generalized by Benallal et al. (1988), belongs to the class of the "return mapping algorithms", presents a quadratic convergence rate, and is quite robust.*

Key words. *Damage, Plasticity, Plates, Fatigue, FEM.*

1. INTRODUCTION

Physically, degradation of the material properties is the result of initiation, growth and coalescence of microcracks or microvoids. In the particular case of isotropic damage there is no dependence of the given plane, with the normal $\vec{n}$, thus, the damage in a point M is given by

$$D(\mathrm{M}) = \frac{\delta S_D}{\delta S} \quad . \tag{1}$$

Let δS_D be the effective area of the intersections of all microcracks or microcavities which lie in δS. The failure is assumed to occur for $D = D_C$, $D_C < 1$, through a process of instability. In agreement with the effective stress concept (Y. N. Robotnov, 1968), considering the 1-D damaged element loaded by a force $\vec{F} = F\vec{n}$, the uniaxial effective stress is given as:

$$\tilde{\sigma} = \frac{F}{S - S_D} \quad . \tag{2}$$

2. DESCRIPTION OF THE PROBLEM

Here, we define the following sets:

Γ_u - is the part of the boundary with prescribed displacement, i.e., $\vec{u} = \bar{\vec{u}}(t)$;

Γ_t - is the part of the boundary with prescribed traction, i.e., $\sigma\vec{n} = \bar{\vec{t}}$;

Ω - is the interior of the domain of the body and

t - is the load control parameter.

2.1. Formulation of the Problem (Thermodynamics and Micromechanics of Damage)

From the Virtual Power Principle we have:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\vec{v})\, d\Omega = \int_\Omega \vec{f} \cdot \vec{v}\, d\Omega + \int_{\Gamma_t} \bar{\vec{t}} \cdot \vec{v}\, d\Gamma \quad . \tag{3}$$

From the Conservation of Energy Principle, for the particular case of isothermal processes, we have

$$\rho \dot{e} = \boldsymbol{\sigma} \cdot \mathbf{D} \quad , \tag{4}$$

where $\dot{e}$ is the specific internal energy rate; ρ is the specific mass and $\mathbf{D}$ is the rate of deformation tensor. In the particular case of isothermic process, the Clausius Duhen inequality reduces to $dS(v)/dt \geq 0 \quad \forall\, v \subset \Omega$. The entropy may be defined as $S(v) = \int_v \rho s dv$, where s is the specific entropy. Form the method of local state equations, we assume the existence of the density of the free energy potential which we denote by Ψ, where:

$$\Psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p, r, \boldsymbol{\alpha}, D) = \Psi(\boldsymbol{\varepsilon}^e, r, \boldsymbol{\alpha}, D) \quad . \tag{5}$$

Here $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$, where $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^p$ are the elastic and plastic part of the strain tensor $\boldsymbol{\varepsilon}$; r is the strain of isotropic hardening; $\boldsymbol{\alpha}$ is the back strain tensor; and D the damage variable. The function $\Psi(\boldsymbol{\varepsilon}^e, r, \boldsymbol{\alpha}, D)$ represent the volume density of free energy potential. This density is defined by $\Psi = e - Ts$. Thus, we derive $\boldsymbol{\sigma} = \rho\, \partial\Psi / \partial\boldsymbol{\varepsilon}$.

We consider the following particular expression for the density of the free energy potential:

$$\rho\Psi = \frac{1}{2}\mathbf{C}\boldsymbol{\varepsilon}^e \cdot \boldsymbol{\varepsilon}^e (1 - D) + R_\infty\left(r + \frac{1}{b}e^{-br}\right) + \frac{1}{3}\chi_\infty \gamma \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} + k_D\left(\frac{D^2}{2} - D_C D\right)\vec{\nabla} D \cdot \vec{\nabla} D \quad , \tag{6}$$

where R_∞ and b are material parameters which characterize the isotropic strain hardening phenomenon; χ_∞ and γ are material parameters which characterize the nonlinear kinematic hardening phenomenon;

$$\mathbf{C} = 2\mu\mathbf{II} + \lambda\mathbf{I}\otimes\mathbf{I} \quad ; \quad \mathbf{II}_{ijrs} = \left(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\right)/2 \quad ; \quad \left(\mathbf{I}\otimes\mathbf{I}\right)_{ijrs} = \delta_{is}\delta_{jr} \quad ; \tag{7}$$

$$2\mu = \frac{E}{1+v} \quad \text{and} \quad \lambda = vE/\left[(1+v)(1-2v)\right] . \tag{8}$$

The Law of elasticity coupled with damage, in the particular case where we have an equal behavior under compression and tension, is given by $\boldsymbol{\sigma} = \rho\,\partial\Psi/\partial\boldsymbol{\varepsilon}^e$. The associated scalar stress is given by $R = \rho\,\partial\Psi/\partial r$. The back stress $\boldsymbol{\chi}^D$ is given by $\boldsymbol{\chi}^D = \rho\,\partial\Psi/\partial\boldsymbol{\alpha}$. The dual variable $\overline{\mathrm{Y}}$, associated with the damage variable D is $\overline{\mathrm{Y}} = \rho\,\partial\Psi/\partial D$. The dual vector $\vec{\Theta}$, associated with the gradient of the damage vector $\vec{\nabla}D$ is given by $\vec{\Theta} = \rho\,\partial\Psi/\partial\vec{\nabla}D$. Where $k_D = \beta\ell_C^{\,2}$, β has the dimension of surface energy, ℓ_C is the characteristic length of the material. For convenience, we define $\mathrm{Y} = -\overline{\mathrm{Y}}$. The elastic strain energy density increment of the material, for the known damage, is

$$d\omega_e = \boldsymbol{\sigma}\, d\varepsilon^e . \tag{9}$$

Thus, replacing $\boldsymbol{\sigma} = \rho\,\partial\Psi/\partial\boldsymbol{\varepsilon}^e$ into Eq. (9) and integrating, we will obtain

$$\mathrm{Y} = \frac{1}{2}\frac{d\omega_e}{dD}\bigg|_{\sigma=cte} + k_D\left(D - D_C\right)\vec{\nabla}D\cdot\vec{\nabla}D . \tag{10}$$

2.2. Kinetic Law of Damage Evolution

In order to describe a dissipative process, we need to introduce complementary kinetic laws. These kinetic laws are obtained by introducing a pseudo-potential of dissipation and by applying to this pseudo-potential the hypothesis of normal dissipation. The unified formulation of Damage Laws assumes the existence of a pseudo potential of dissipation $F\left(\boldsymbol{\sigma}, R, \boldsymbol{\chi}^D, \mathrm{Y}; \boldsymbol{\varepsilon}^e, r, \boldsymbol{\alpha}, D, \vec{\nabla}D\right) \to \Re$. It is a scalar continuous function, convex with respect to the variable $\left(\boldsymbol{\sigma}, R, \boldsymbol{\chi}^D, \mathrm{Y}\right)$. The complementary laws, which describe the dissipative process, are then derived from this potential by the normality rule of generalized standard materials. These complementary equations are given by the following set of evolution laws:

- Plastic evolution equations

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\sigma}} \quad , \quad \dot{r} = -\dot{\lambda}\frac{\partial F}{\partial R} \quad , \quad \dot{\boldsymbol{\alpha}} = -\dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\chi}^D} \; ; \tag{11}$$

- Damage evolution equation

$$\dot{D} = \dot{\lambda}\frac{\partial F}{\partial \mathrm{Y}} . \tag{12}$$

2.3. Constitutive Equation of the Cycle Elastoplastic with Damage

The loading function is given by:

$$f=\left(\tilde{\boldsymbol{\sigma}}^D-\boldsymbol{\chi}^D\right)_{eq}-R-\sigma_y \ , \ \text{with} \quad \left(\tilde{\boldsymbol{\sigma}}^D-\boldsymbol{\chi}^D\right)_{eq}=\left[\frac{3}{2}\left(\tilde{\boldsymbol{\sigma}}^D-\boldsymbol{\chi}^D\right)\cdot\left(\tilde{\boldsymbol{\sigma}}^D-\boldsymbol{\chi}^D\right)\right]^{1/2} . \quad (13)$$

The accumulated plastic strain $\dot{p}$ is such that $\dot{p}=\dot{\lambda}/(1-D)$. The evolution laws for $\dot{R}$, $\dot{\boldsymbol{\chi}}^D$, $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{D}$ may be written in an incremental form, by integrating these rate equations between t_n and t_{n+1}. In this case, we have:

$$R_{n+1}=R_n+\Delta R \ , \quad \boldsymbol{\chi}_{n+1}^D=\boldsymbol{\chi}_n^D+\Delta\boldsymbol{\chi}^D \ , \quad D_{n+1}=D_n+\Delta D \quad \text{and} \quad \boldsymbol{\varepsilon}_{n+1}^p=\boldsymbol{\varepsilon}_n^p+\Delta\boldsymbol{\varepsilon}^p \ ; \quad (14)$$

where $\Delta R=b\left(R_\infty-R_{n+\theta}\right)\Delta\lambda$, $R_{n+\theta}=(1-\theta)R_n+\theta R_{n+1} \quad \theta\in(0,1)$, (15)

$$\Delta\boldsymbol{\chi}^D=-\gamma\left[\boldsymbol{\chi}_{n+\theta}^D-\frac{\chi_\infty}{\left(\tilde{\boldsymbol{\sigma}}^D-\boldsymbol{\chi}^D\right)_{n+\theta}}\left(\tilde{\boldsymbol{\sigma}}^D-\boldsymbol{\chi}^D\right)_{n+\theta}\right]\Delta\lambda \ , \quad (16)$$

$$\Delta D=\frac{\Upsilon_{n+\theta}H\left(p_{n+\theta}-p_D\right)}{S}\Delta p \ , \quad H(p-p_D)=\begin{cases}1 & \text{se} \quad p\geq p_D\\ 0 & \text{se} \quad p<0\end{cases} \ , \quad (17)$$

$$\Delta p\left(1-D_{n+\theta}\right)=\Delta\lambda \ , \quad \Delta\boldsymbol{\varepsilon}^p=\frac{3}{2}\frac{\left(\tilde{\boldsymbol{\sigma}}_{n+\theta}^D-\boldsymbol{\chi}_{n+\theta}^D\right)}{\left(\tilde{\boldsymbol{\sigma}}_{n+\theta}^D-\boldsymbol{\chi}_{n+\theta}^D\right)_{eq}}\frac{\Delta\lambda}{\left(1-D_{n+\theta}\right)} \quad \text{and} \quad (18)$$

$$p_{n+1}=p_n+\Delta p \quad \text{thus} \quad p_{n+1}=p_n+\frac{\Delta\lambda}{\left(1-D_{n+\theta}\right)} \ . \quad (19)$$

The two additional equations that we must impose are:

- The elastic strain versus stress constitutive eqn. at t_{n+1}

$$\boldsymbol{\varepsilon}_{n+1}^e=\left(1-D_{n+1}\right)^{-1}\mathbf{C}^{-1}\boldsymbol{\sigma}_{n+1} \quad \text{where} \quad \mathbf{C}^{-1}=(2\mu)^{-1}\mathbf{II}-\left[(\lambda/2\mu)/(2\mu+3\lambda)\right]\mathbf{I}\otimes\mathbf{I} \ . \quad (20)$$

- The von Mises yield criterion

$$f=\left(\tilde{\boldsymbol{\sigma}}_{n+1}^D-\boldsymbol{\chi}_{n+1}^D\right)_{eq}-R_{n+1}+\sigma_Y \ , \quad \text{with} \quad (21)$$

$$\left(\tilde{\boldsymbol{\sigma}}_{n+1}^D-\boldsymbol{\chi}_{n+1}^D\right)_{eq}=\left[\frac{3}{2}\left(\tilde{\boldsymbol{\sigma}}_{n+1}^D-\boldsymbol{\chi}_{n+1}^D\right)\cdot\left(\tilde{\boldsymbol{\sigma}}_{n+1}^D-\boldsymbol{\chi}_{n+1}^D\right)\right]^{1/2} . \quad (22)$$

2.4. Set of nonlinear equations

Let $\vec{q}=\left(\Delta\lambda, R_{n+1}, \chi_{n+1}^D, D_{n+1}, \vec{\nabla}D_{n+1}, \boldsymbol{\sigma}_{n+1}, \varepsilon_{n+1}^p\right)$. The objective is to det. $\vec{q}$ so that the following set of equations. Is satisfied:

$$g_i(\vec{q})=0 \quad , \quad i=1,\ldots,6 \quad (23)$$

where $g_1(\vec{q})=\left(\tilde{\boldsymbol{\sigma}}_{n+1}^D-\boldsymbol{\chi}_{n+1}^D\right)_{eq}-R_{n+1}-\sigma_y$, $\quad g_2(\vec{q})=\boldsymbol{\varepsilon}_{n+1}^e-\left[\mathbf{C}^{-1}/\left(1-D_{n+1}\right)\right]\boldsymbol{\sigma}_{n+1}$,

$$g_3(\vec{q})=R_{n+1}-R_n-\Delta R \ , \qquad g_4(\vec{q})=\boldsymbol{\chi}_{n+1}^D-\boldsymbol{\chi}_n^D-\Delta\boldsymbol{\chi}^D \ ,$$

$$g_5(\vec{q})=D_{n+1}-D_n-\Delta D \quad \text{and} \quad g_6(\vec{q})=\boldsymbol{\varepsilon}_{n+1}^p-\boldsymbol{\varepsilon}_n^p-\Delta\boldsymbol{\varepsilon}^p \ . \quad (24)$$

3. THICK PLATE THEORY

The classic theories (of first order) of plates presented by (Timoshenko and Woinowsky-Krieger, 1959) and by (Reissner, 1945; Mindlin, 1951) are not able to describe properly the plastification of the cross section of thick plates. With the objective of improving the description of the elastoplastic with damage front on the cross section, we make use of higher order theory (Costa Jr., 1998).

The theory incorporates a displacement field with the following characteristics:

- Quadratic variation of the transversal shear stress;
- Linear variation of the normal deformation;
- Consideration of the three dimensional Hooke Law for the elastic constitutive equation.

The field of displacement is given by $\vec{u} = u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z$, being $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ the unitary vectors that form the base of the global Cartesian system and

$$u_x(x,y,z) = u(x,y) + z\theta_y(x,y) + z^3\theta^*_y(x,y),$$
$$u_y(x,y,z) = v(x,y) - z\theta_x(x,y) - z^3\theta^*_x(x,y),$$
$$u_z(x,y,z) = w(x,y) + z^2 w^*(x,y) \;; \qquad (25)$$

as a result, the infinitesimal strain tensor is given as:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \;;\quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \;;\quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \;; \qquad (26)$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \;;\quad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \quad \text{and} \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\;. \qquad (27)$$

These components may be written as (Batoz *et al.*, 1992):

$$\varepsilon_{xx} = e^o_{xx} + z\aleph^o_{xx} + z^3\aleph^*_{xx} \;;\quad \varepsilon_{yy} = e^o_{yy} + z\aleph^o_{yy} + z^3\aleph^*_{yy} \;;\quad \varepsilon_{zz} = z e_{nn} \;; \qquad (28)$$

$$\gamma_{xy} = \left(e^o_{xy} + e^o_{yx}\right) + z\left(\aleph^o_{xy} + \aleph^o_{yx}\right) + z^3\left(\aleph^*_{xy} + \aleph^*_{yx}\right) \;;$$

$$\gamma_{xz} = \gamma^o_{xz} + z^2\gamma^*_{zx} \quad \text{and} \quad \gamma_{yz} = \gamma^o_{yz} + z^2\gamma^*_{zy}\;. \qquad (29)$$

Integrating the first member of the Eq. (3) in z, $z \in \left(-\frac{h}{2}, \frac{h}{2}\right)$, h - the thickness of the plate, we obtain:

$$\int_A \{N\}^T \cdot \{e^o\} + \{M\}^T \cdot \{\aleph^o\} + \{M^*\}^T \cdot \{\aleph^*\} + \{Q\}^T \cdot \{\gamma^o\} + \{Q^*\}^T \cdot \{\gamma^*\} + N_{nn} e_{nn} dA \;. \qquad (30)$$

The generalized loads, associated with the theory, can be expressed as:

- The membrane loading, given in force by unit of length (width).

$$\{N\} = \begin{Bmatrix} N_{xx} \\ N_{xy} \\ N_{yx} \\ N_{yy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yx} \\ \sigma_{yy} \end{Bmatrix} dz \;; \qquad (31)$$

- The bending for unit of length (width).

$$\{M\} = \begin{Bmatrix} M_{xx} \\ M_{xy} \\ M_{yx} \\ M_{yy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yx} \\ \sigma_{yy} \end{Bmatrix} z\, dz \quad \text{and} \quad \{M^*\} = \begin{Bmatrix} M^*_{xx} \\ M^*_{xy} \\ M^*_{yx} \\ M^*_{yy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yx} \\ \sigma_{yy} \end{Bmatrix} z^3 dz \;; \qquad (32)$$

- The shear loading for unit length (width).

$$\{Q\}=\begin{Bmatrix}Q_x\\Q_y\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_{xz}\\\sigma_{yz}\end{Bmatrix}z\,dz \quad \text{and} \quad \{Q^*\}=\begin{Bmatrix}Q_x^*\\Q_y^*\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_{xz}\\\sigma_{yz}\end{Bmatrix}z^2dz\ ; \tag{33}$$

- The normal loading for unit length (width).

$$N_{nn}=\int_{-h/2}^{h/2}\sigma_{zz}z\,dz\ . \tag{34}$$

Integrating the first part of the second member of the Eq. (3) in z, $z\in\left(-h/2, h/2\right)$, considering that $\vec{f}=\rho\vec{g}=cte$, $\vec{g}$ - the vectorial acceleration of the gravity, we obtain:

$$\int_{\mathrm{A}}\rho\vec{g}\cdot\vec{v}\,d\mathrm{A}\ , \tag{35}$$

for the second part of the second member, we derive:

$$\int_{\Gamma_t}\vec{\bar{t}}.\vec{v}\,d\Gamma=\int_{\mathrm{S}}\left\{\overline{N}\right\}\cdot\left\{u^o\right\}+\left\{\overline{M}\right\}\cdot\left\{\theta^o\right\}+\left\{\overline{N}^*\right\}\cdot\left\{u^*\right\}+\left\{\overline{M}^*\right\}\cdot\left\{\theta^*\right\}d\mathrm{S} \tag{36}$$

$$\text{where}\ \left\{\hat{u}^o\right\}=\begin{Bmatrix}\hat{u}\\\hat{v}\\\hat{w}\end{Bmatrix},\quad \left\{\hat{\theta}^o\right\}=\begin{Bmatrix}\hat{\theta}_y\\-\hat{\theta}_x\\0\end{Bmatrix},\quad \left\{\hat{u}^*\right\}=\begin{Bmatrix}\hat{u}^*\\\hat{v}^*\\\hat{w}^*\end{Bmatrix}\quad \text{and}\quad \left\{\hat{\theta}^*\right\}=\begin{Bmatrix}\hat{\theta}_y^*\\-\hat{\theta}_x^*\\0\end{Bmatrix}\ ; \tag{37}$$

$$\left(\overline{N}\quad \overline{M}\quad \overline{N}^*\quad \overline{M}^*\right)=\int_{-h/2}^{h/2}\left(1\quad z\quad z^2\quad z^3\right)\vec{\bar{t}}\,dz\,. \tag{38}$$

4. ALGORITHM

In this section we present a return mapping algorithm, already exposed by Rossi & Alves (1996), that enable us to integrate the fully coupled elastoplastic with damage constitutive equations. This proposed algorithm decomposes the problem into two parts. The former consists in the global equilibrium equations and is showed in the finite element formulation and the second deals with the local integration of the constitutive equations. However, in order to perform the integration of the global equilibrium equations we must derive the consistent tangent operator $\mathbf{J}$.

4.1. Global Equilibrium Equations

The basic problem in nonlinear analysis is based (Bathe, 1982), in general, on the consideration of the equilibrium configuration at time t_{n+1}. The external forces, $\vec{F}e_{n+1}$, are generally "time" dependent and the equilibrium state can be written as $\vec{F}e_{n+1}-\vec{F}i_{n+1}=0$, where $\vec{F}i_{n+1}$ is the internal forces.

With the objective of solving the nonlinear equations associated with our problem, we make use of the Newton-Raphson method. Now, and formulate the problem as:

$$\vec{h}\left(\vec{u}^*\right)=0\ , \tag{39}$$

where $\vec{h}\left(\vec{u}^*\right)=\vec{F}e_{n+1}\left(u^*\right)-\vec{F}i_{n+1}\left(\vec{u}^*\right)=0$, then this method consist in the expansion of $\vec{h}\left(\vec{u}^*\right)$ in a Taylor series where we keep only the first order term. Moreover, if we denote

$$\Delta\vec{u}^{i}_{n+1} = \left(\vec{u}^{*} - \vec{u}^{i}_{n+1}\right) \quad \text{and} \quad \mathbf{K}^{i}_{n+1} = \frac{\partial \vec{F}i\left(\vec{u}^{i}_{n+1}\right)}{\partial \vec{u}} \ , \tag{40}$$

the method may be expressed as

$$\mathbf{K}^{i}_{n+1}\Delta\vec{u}^{i}_{n+1} = \vec{F}e_{n+1} - \vec{F}i^{i}_{n+1} \ , \tag{41}$$

where $\mathbf{K}$ is the tangent stiffness matrix. The update procedure is then given by

$$\vec{u}^{i+1}_{n+1} = \Delta\vec{u}^{i}_{n+1} + \vec{u}^{i}_{n+1} \ . \tag{42}$$

Here, we adopt the following convergence criteria: $\left\|\vec{h}\left(\vec{u}^{i+1}_{n+1}\right)\right\| \leq tolerance$, resulting $\vec{u}^{i+1}_{n+1} \approx \vec{u}^{*}$ at convergence. Based on an incremental approach, the tangent stiffness can be written as

$$\mathbf{K}^{i}_{n+1} = \int_{\Omega} \mathbf{B}^{T}\mathbf{J}^{i}_{n+1}\mathbf{B}\, d\Omega \ , \tag{43}$$

where $\mathbf{B}$ is the strain-displacement matrix given by:

$$\mathbf{B} = \left[\mathbf{I}_{e^o\aleph}\right]\left\{e^{o}\right\} + z\left[\mathbf{I}_{e^o\aleph}\right]\left\{\aleph^{o}\right\} + z^{3}\left[\mathbf{I}_{e^o\aleph}\right]\left\{\aleph^{*}\right\} + z\left[\mathbf{I}_{e_{nn}}\right]e_{nn} + \left[\mathbf{I}_{\gamma}\right]\left\{\gamma^{o}\right\} + z^{2}\left[\mathbf{I}_{\gamma}\right]\left\{\gamma^{*}\right\} \ , \tag{44}$$

where

$$\left[\mathbf{I}_{e^o\aleph}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \quad \left[\mathbf{I}_{\gamma}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \quad \left[\mathbf{I}_{e_{nn}}\right] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \tag{45}$$

$$\left\{e^{o}\right\} = \begin{Bmatrix} e^{o}_{xx} \\ e^{o}_{xy} \\ e^{o}_{yx} \\ e^{o}_{yx} \end{Bmatrix} = \left[\mathbf{B}_{e^o}\right]\vec{q}_{e} \ , \quad \left\{\aleph^{o}\right\} = \begin{Bmatrix} \aleph^{o}_{xx} \\ \aleph^{o}_{xy} \\ \aleph^{o}_{yx} \\ \aleph^{o}_{yy} \end{Bmatrix} = \left[\mathbf{B}_{\aleph^o}\right]\vec{q}_{e} \ , \quad \left\{\aleph^{*}\right\} = \begin{Bmatrix} \aleph^{*}_{xx} \\ \aleph^{*}_{xy} \\ \aleph^{*}_{yx} \\ \aleph^{*}_{yy} \end{Bmatrix} = \left[\mathbf{B}_{\aleph^*}\right]\vec{q}_{e} \ , \tag{46}$$

$$e_{nn} = 2w^{*} = \left\{\mathbf{B}_{\aleph^*}\right\}\cdot\vec{q}_{e} \ , \quad \left\{\gamma^{o}\right\} = \begin{Bmatrix} \gamma^{o}_{xz} \\ \gamma^{o}_{yz} \end{Bmatrix} = \left[\mathbf{B}_{\gamma^o}\right]\vec{q}_{e} \ , \quad \left\{\gamma^{*}\right\} = \begin{Bmatrix} \gamma^{*}_{xz} \\ \gamma^{*}_{yz} \end{Bmatrix} = \left[\mathbf{B}_{\gamma^*}\right]\vec{q}_{e} \tag{47}$$

and $\mathbf{J}$ the tangent operator. In the first iteration we consider $\vec{u}^{0}_{n+1} \approx \vec{u}^{*}_{n}$.

4.2. Local Integration

In order to simplify the notation while describing the algorithm we introduce two vectors: $\vec{Q} = \left(\boldsymbol{\varepsilon}^{p}, \boldsymbol{\chi}^{D}, R, D\right)$ and $\vec{q} = \left(\boldsymbol{\sigma}, \vec{Q}, \lambda\right)$. Thus, the evolution equations early presented can be written in a compact form as

$$\dot{\vec{Q}} = \dot{\lambda}\vec{G}\left(\boldsymbol{\sigma}, \vec{Q}\right) \ . \tag{48}$$

Making use of an incremental form, we obtain $\Delta\vec{Q} - \Delta\lambda\vec{G}\left(\boldsymbol{\sigma}_{n+\theta}, \vec{Q}_{n+\theta}\right) = 0$, where $\Delta(\circ)=(\circ)_{n+1}-(\circ)_n$ and $(\circ)_{n+\theta}=(1-\theta)(\circ)_n+\theta(\circ)_{n+1}$ as $\theta\in[0,1]$.

(i) Given $\boldsymbol{\varepsilon}_{n+1}$, we assume a purely elastic increment, thus $\Delta Q=0$ and $\Delta\lambda=0$. The trial tension is then computed by $\boldsymbol{\sigma}=\mathbf{C}\boldsymbol{\varepsilon}^*$, where $\boldsymbol{\varepsilon}^*=\boldsymbol{\varepsilon}_{n+1}-\boldsymbol{\varepsilon}_n^p$.

(ii) With the trial tension in *(i)* we check the yield function. If $f\left(\tilde{\boldsymbol{\sigma}}_{n+1},\vec{Q}_n\right)<0$ then hypothesis *(i)* is correct and the local procedure is complete with no updating in the remaining components of $\vec{q}$. However, if $f\left(\tilde{\boldsymbol{\sigma}}_{n+1},\vec{Q}_n\right)\geq 0$ then we must perform an elastoplastic with damage correction.

(iii) In order to perform the plastic corrections, we employ the incremental equations. Now, in the incremental procedure we must satisfy at time t_{n+1} the yield criteria, the elasticity law and the evolution equations. With these three conditions we derive the following set of nonlinear equations (Eqns. (24)):

$$g_1(\vec{q}_{n+1})=f\left(\tilde{\boldsymbol{\sigma}}_{n+1},\vec{Q}_{n+1}\right)=0\ ,\qquad g_{2\ldots5}(\vec{q}_{n+1})=\Delta\vec{Q}-\Delta\lambda\vec{G}\left(\boldsymbol{\sigma}_{n+\theta},\vec{Q}_{n+\theta}\right)=0\qquad\text{and}$$
$$g_6(\vec{q}_{n+1})=\boldsymbol{\varepsilon}_{n+1}^e-\mathbf{C}^{-1}\tilde{\boldsymbol{\sigma}}\tag{49}$$

where $\boldsymbol{\varepsilon}_{n+1}^e=\boldsymbol{\varepsilon}_{n+1}-\boldsymbol{\varepsilon}_{n+1}^p$.

This system is also solved by the Newton-Raphson method, i.e.

$$\mathbf{M}_{\mathbf{T}}^k\vec{\Delta}q_{n+1}^k=-\vec{g}\left(\vec{q}_{n+1}^k\right)\ ,\tag{50}$$

where $\mathbf{M}_{\mathbf{T}ij}^{\ k}=\partial g_i/\partial q_j$ and $\vec{q}_{n+1}^0=\left(\boldsymbol{\sigma}_n,\vec{Q}_n,0\right)$. The update is done through $\vec{q}_{n+1}^{k+1}=\vec{q}_{n+1}^k+\vec{\Delta}q_{n+1}^k$ and convergence criteria is given by $\left\|\vec{g}_{n+1}\right\|\leq tolerance$.

4.3. Determination of the Tangent Operator $\mathbf{J}$

When the local integration algorithm has converged, the corresponding consistent operator associated with the discretization may be determined by letting all the variables $\vec{q}$ and $\boldsymbol{\varepsilon}$ vary slightly around the solution at the converged solution at iteration $n+1$. Thus,

$$\delta\boldsymbol{\sigma}=\mathbf{J}\delta\boldsymbol{\varepsilon},\quad\text{with}\quad \mathbf{J}_{n+1}^i=\left.\frac{\partial\boldsymbol{\sigma}(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}_n,D_n,\boldsymbol{\varepsilon}_n^p,\boldsymbol{\varepsilon}(\vec{u})-\boldsymbol{\varepsilon}_n)}{\partial\boldsymbol{\varepsilon}}\right|_{\vec{u}_{n+1}^i}\ .\tag{51}$$

Now, in order to compute the above differentiation we must enforce the evolution equations. These evolution equations may be written in an incremental form as:

$$\sum_{i=1}\left[\left(\frac{\partial g_i}{\partial q_j}\right)_{n+1}\delta q_j+\left(\frac{\partial g_i}{\partial\boldsymbol{\varepsilon}}\right)_{n+1}\delta\boldsymbol{\varepsilon}\right]=0\qquad j=1,\ldots,6\ .\tag{52}$$

From these considerations we determine:

$$\int_\Omega\mathbf{C}_{n+1}^i\boldsymbol{\varepsilon}(\Delta\vec{u}_{n+1}^i)\cdot\boldsymbol{\varepsilon}(\vec{v})\ d\Omega\ .\tag{53}$$

The stiffness matrix can be finally obtained by integrating Eq. (43) in z, $z\in\left(-\tfrac{h}{2},\tfrac{h}{2}\right)$.

5. EXEMPLE

In this section we solve the problem consisting of a plate with thickness h=28mm, width b=100mm, length ℓ=200mm as illustrated in Fig. 1 with both ends clamped and subjected, at the mid span, to a cycle prescribed displacement. The prescribed displacement field where $\bar{u}(t)=\Delta\bar{u}\sin(2\pi t)$,with $\Delta\bar{u}$=3.3mm. The material properties used in this work are: Young Modulus E=200,000MPa, Poisson's ratio ν=0.3, σ_y=260MPa, χ_∞=200MPa, b=1, γ=2, R_∞=300Mpa, and D_c=0.15. Moreover, k_D=0.0 MPam^2 and S=7.0Mpa. Here, t denotes a loading parameter.

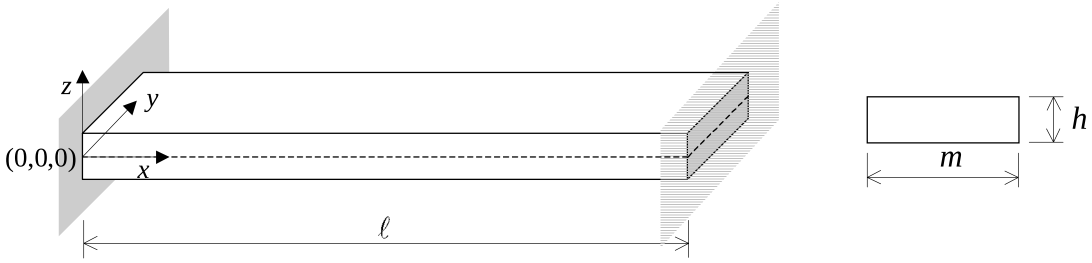

Figure 1 – Geometric illustration of plate.

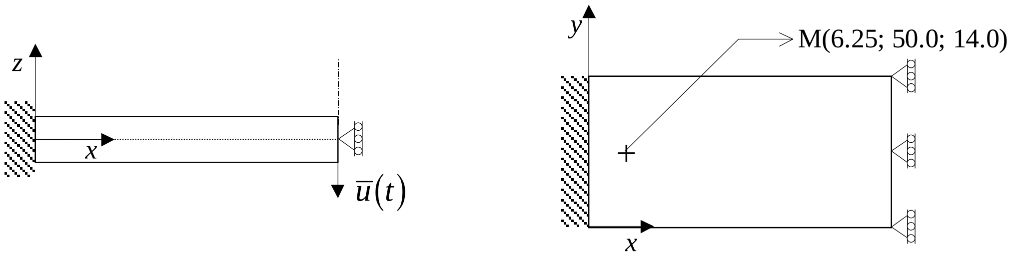

Figure 2 – Physical problem (using symmetry).

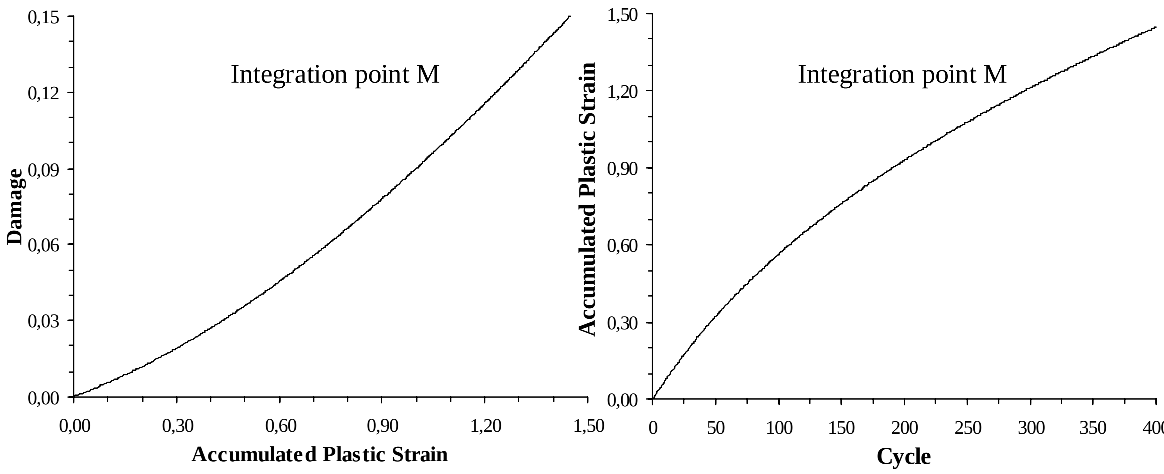

Figure 3 – Nonlinear evolution.

Here we employed the Quad9 isoparametric element, Dhatt *et al.* (1984). The integration scheme used was 3 x 3 x 9 integration, 9 Newton-Cotes points through the thickness, and it was 3 x 3 Gauss points in ξ and η co-ordinates.

Figure 4 – Damage evolution.

6. CONCLUSION

The theoretical approach used in this work allow us to consider a very broad class of problems with ability to model quite complex phenomena. Moreover, since low cycle fatigue is strongly related to the accumulated plastic strain, and since we are able to the model properly quite complex cyclic plastic phenomena under general loading conditions, we can determine with some accuracy the critical points and the number of cycles that a given component may withstand before there is a mesocrack nucleation. However, due to the considerable large number of iterations necessary to perform a low cycle fatigue analysis we must have an accurate algorithm, with a very high rate of convergence. The proposed algorithm has accomplished this objective. The disadvantage of this approach is that once the critical damage is reached, we must in order to continue the analysis employ the fracture mechanics method.

REFERENCES

Bathe, K. J. – Finite Elements Procedures in Engineering Analysis – New Jersey, prentice-Hall, 1982.

Batoz, Jean-Louis & Dhatt, Gouri – Modélisation des Structures par Éléments Finis – vol. 3, Paris, Hermes, 1992.

Benallal, A.; Billardon, R. & Doghri, I. – An Integration Algorithm and the Corresponding Consistent Tangent Operator for Fully Coupled Elastoplastic and Damage Equations – Communications in Applied Numerical Methods, vol. 4, pp. 731-740, 1988.

Costa Jr., J. C. A. – Análise Numérica do Dano em Placas espessas sob Fadiga de Baixo Ciclo – Dissertação de Mestrado, UFSC, 1998.

Dhatt G. & Touzot G. – The Finite Element Method Displayed – John Wiley and Sons, 1984, ISBN 0-471-90110-5.

Lemaitre, J. – A Course on Damage Mechanics – Germany, Springer-Verlag, p. 209, 1992.

Robotnov, Y. N. – Creep Rupture – Proc. XII, Inter. Congress Appl. Mech., Stanford, Springer Berlin, 1969.

Rossi, R., Alves, M. K. – Análise do Dano de Fadiga de Baixo Ciclo Considerando o Efeito do Fechamento de Microtrincas – Anais do IX Sibrat, Rio de Janeiro, 1996.

Simo, J. C. & Taylor, R. L. – A Return Mapping Algorithm for Plane Stress Elastoplasticity – International Journal for Numerical Methods in Engineering, vol. 31, pp. 649-670, 1986.